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Infinitesimal weak symmetries of nonlinear differential equations in two independent variables

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Abstract. Non-classical infinitesimal weak symmetries of PDE introduced by Olver and Rosenau are analysed. In the case of PDE in two independent variables, it is demonstrated that obtaining such symmetries is equivalent to obtaining the two-dimensional modules of non-classical partial symmetries. The Boussinesq and the nonlinear heat equations are treated from the point of view of non-classical symmetries.

1. Introduction

The classical Lie symmetries provide a rich variety of methods for analysing and solving differential equations. One can try to obtain invariant, partially invariant solutions and conservation laws to perform a group foliation, to transform a given nonlinear system to a less complicated or linear system via contact transformations, etc [1–4]. The common feature of these procedures is a reduction of the original problem to a simpler or well known one. For example, obtaining invariant solutions reduces to solving quotient differential equations in fewer independent variables than the original equations. In particular, these quotient equations might be ordinary differential or algebraic equations. For this reason, finding invariant solutions is the most popular application of the Lie symmetries.

Since Sophus Lie, several new types of non-classical symmetries of differential equations, aimed at generalizing the concept of invariant solutions, have been proposed. We restrict ourselves to the class of conditional non-classical symmetries. The term 'conditional' is explained by the fact that symmetries of a new system of differential equations are examined. The latter is obtained by appending additional differential equations, called *side conditions*, to the original system.

Bluman and Cole [5] considered the two-dimensional linear heat equation $u_t = u_{xx}$ attached by the first-order differential equation which was the necessary and sufficient condition of invariance of the functions under a certain vector field in the space R^3 of the variables t, x, u. The vector field was taken to be a classical infinitesimal symmetry of the system. The symmetries of the Bluman and Cole type drew considerable attention later in [6–10]. One of the reasons of interest is their connection with the direct method of Clarkson and Kruskal for obtaining explicit solutions of nonlinear differential equations [11–14]. Specifically, all known solutions obtained by the direct method are invariant under the non-classical conditional symmetries [15].

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The Bluman and Cole approach was generalized in [16, 17], where involutive modules of vector fields of contact infinitesimal symmetries were considered. They were called *partial symmetries*. It was demonstrated that the modules of partial symmetries were closely related to the differential substitutions of the Hopf–Cole type, the Bäcklund transformations, functionally invariant solutions of Smirnov and Sobolev, and so on. Note also that in [16] the side conditions selecting the class of partially invariant solutions and corresponding non-classical symmetries were constructed. Fushchich *et al* [18] proposed associating to differential equations admitting the classical symmetry groups additional differential equations which were differential invariants of the classical Lie symmetry group and which were compatible with the original equations, and to find the classical Lie symmetries of the associated system. It is evident that in general the group thus obtained is an extension of the classical group of the original differential equation. Olver and Rosenau [19, 20] considered a new type of non-classical symmetries. Their weak symmetries were defined as groups G of transformations such that G-invariant solutions could be obtained from the reduced equations in fewer independent variables. Several examples of one-parameter weak symmetry groups and of the corresponding invariant solutions were given in [19, 20]. It is clear that the proposed non-classical symmetries need further investigation. In particular, their interrelationship, scopes of applicability, and connection to the classical Lie symmetries are of great interest.

The present paper is mainly devoted to weak and partial symmetries. In section 2 elements of theory of partial symmetries for differential equations in one unknown function is developed. Theorem 3 of section 3 demonstrates that obtaining weak symmetries for differential equations in two independent variables is equivalent in a generic case to obtaining two-dimensional modules of partial symmetries, which is much easier. Partial symmetries and corresponding invariant solutions of the family of nonlinear heat equations are considered in section 4. Section 5 reveals that there are some interesting special cases, where finding weak symmetries does not fall under theorem 3 of the paper.

2. Non-classical partial symmetries

In [16, 17] partial symmetries were developed for systems of differential equations. The differential equations in one unknown function need separate consideration (given below), since, by the Bäcklund theorem, the structure of contact transformations depends on the number (one or more) of unknown functions. Therefore, we consider a nonlinear differential kth-order equation

$$\Delta(t, x, u, p) = 0 \tag{1}$$

for the real-valued function $u(t, x)$. In (1) t and x are independent variables, u is a dependent variable, p is a set of symbols of partial derivatives p_σ of the function $u(t, x)$ of orders $1 \leqslant |\sigma| \leqslant k$. Here $\sigma = (i_1, i_2)$ is a multi-index, $|\sigma| = i_1 + i_2$, and the variable p_σ corresponds to the partial derivative $\partial^{|\sigma|}u(t, x)/\partial t^{i_1}\partial x^{i_2}$. The variables t, x, u and p are regarded as coordinates of points belonging to the space $J^k(R^3)$, where R^3 consists of the points (t, x, u). Let $f(t, x, u, p_t, p_x)$ be an arbitrary smooth function defined on $J^1(R^3)$ with $p_t \equiv p_{(1,0)}$, $p_x \equiv p_{(0,1)}$. Consider the contact vector field

$$X_f = -f_{p_t}\partial t - f_{p_x}\partial x + (f - p_t f_{p_t} - p_x f_{p_x})\partial u + (f_t + p_t f_u)\partial p_t + (f_x + p_x f_u)\partial p_x \tag{2}$$

on the space $J^1(R^3)$. The function f is called a *characteristic function* of the vector field X_f. The vector field X_f is a classical infinitesimal (tangent) symmetry of equation (1), treated as a hypersurface E_Δ in $J^k(R^3)$, if E_Δ is an invariant submanifold under the prolongation $X_f^{(k)}$ of the vector field X_f to the space $J^k(R^3)$. The relation $X_f^{(k)}(\Delta)(t,x,u,p) = A(t,x,u,p)\Delta(t,x,u,p)$ serves as an infinitesimal criterion of the invariance with some function $A(t,x,u,p)$ on $J^k(R^3)$ [1–4].

The functions $u = u(t,x)$ invariant under X_f are the only functions satisfying the following first-order differential equation:

$$f(t,x,u,p_t,p_x) = 0. \tag{3}$$

Denote by E_f a submanifold of $J^1(R^3)$ determined by equation (3), and denote by $E_f^{(k)}$ its prolongation to $J^k(R^3)$. The submanifold $E_f^{(k)}$ is given by (3) and by the equations

$$D_\mu(f) = 0 \qquad |\mu| \leqslant k-1 \tag{4}$$

appended to (3), where $\mu = (j_1, j_2)$ is a multi-index, $D_\mu = D_t^{j_1} \circ D_x^{j_2}$,

$$\begin{aligned} D_t &= \partial t + p_t \partial u + \ldots + p_{\sigma+1_1}\partial p_\sigma + \ldots \\ D_x &= \partial x + p_x \partial u + \ldots + p_{\sigma+1_2}\partial p_\sigma + \ldots \end{aligned} \tag{5}$$

are operators of variational derivatives, and $\sigma + 1_j$ is the multi-index obtained from σ by adding unity to its jth component. It is easy to check that the vector field X_f is tangent to E_f, and $X_f^{(k)}$ *is tangent to* $E_f^{(k)}$.

Below we assume that the rank $\|f_{p_t}, f_{p_x}\| = 1$. The vector field X_f is known as a nonclassical partial symmetry of equation (1) if $X_f^{(k)}$ is tangent to the intersection $E_\Delta \cap E_f^{(k)}$. In the case when f is linear in the variables p_t, p_x or, in other words, in the case when X_f is obtained by prolongation from R^3, the symmetries just defined were introduced in [5]. The infinitesimal criterion for X_f to be the partial symmetry is the relation

$$\begin{aligned} &X_f^{(k)}(\Delta)(t,x,u,p) = A(t,x,u,p)\Delta(t,x,u,p) + B_\mu(t,x,u,p)D\mu(f) \\ &\qquad\qquad 0 \leqslant |\mu| \leqslant k-1 \end{aligned}$$

where A, B_μ are some functions on $J^k(R^3)$.

Theorem 1. Let X_f be an infinitesimal partial symmetry of equation (1); then for each function $\widetilde{f}$, which regularly determines the submanifold E_f, the vector field $X_{\widetilde{f}}$ is also an infinitesimal partial symmetry. Further, if $\widetilde{f} = gf$, then $X_{\widetilde{f}}^{(k)}\big|_{E_f^{(k)}} = gX_f^{(k)}$.

Proof. The vector field X_f given by formula (2) may be rewritten in the form

$$X_f = -f_{p_t}D_t - f_{p_x}D_x + f\partial u + D_t(f)\partial p_t + D_x(f)\partial p_x \tag{6}$$

where $D_t = \partial t + p_t\partial u$ and $D_x = \partial x + p_x\partial u$. Its prolongation to $J^k(R^3)$ may be given by the formula

$$X_f^{(k)} = X_f + D_\mu(f)\partial p_\mu \qquad |\mu| \leqslant k \tag{7}$$

with D_μ given by formulae (5). The assertion of the theorem follows immediately from (6), (7), (3) and (4). □

Corollary. If the relation rank $\|f_{p_t}, f_{p_x}\| = 1$ is valid, the characteristic function of the infinitesimal partial symmetry can be chosen in the form $f = -p_t + a(t, x, u, p_x)$ or in the form $f = -p_x + b(t, x, u, p_t)$.

For the partial infinitesimal symmetry X_f, the problem of obtaining the solution of equation (1) invariant with respect to X_f reduces to the solution of an ODE, provided the intersection $E_\Delta \cap E_f^{(k)}$ is non-empty. This reduction is based on the fact that each solution $u = u(t, x)$ of equation (3) is invariant under X_f. Therefore, it is uniquely determined by its values $v(s) = u\big(\alpha(s), \beta(s)\big)$ on the curve $\gamma : t = \alpha(s), x = \beta(s)$ such that one can express the derivatives u_t, u_x on γ through $\dot{v}(s)$ from the equations $\dot{v}(s) = u_t\dot{\alpha}(s) + u_x\dot{\beta}(s)$ and from (3) restricted to γ. In that case X_f is said to be transversal to γ. If the function $v(s)$ satisfies the quotient equation which is one of the equations of the restriction of $E_\Delta \cap E_f^{(k)}$ on γ, then a unique solution of the Cauchy problem for equation (3) with $v(s)$ as an initial datum simultaneously satisfies equation (1) [17].

Now let us consider a pair of contact vector fields X_f and X_g with functionally independent characteristic functions f and g and try to figure out when the reduction of equation (1) to an algebraic equation is possible for obtaining solutions invariant under X_f and X_g. Let $E_{f,g}$ be a submanifold determined by the pair of equations

$$f(t, x, u, p_t, p_x) = 0 \qquad g(t, x, u, p_t, p_x) = 0 \tag{8}$$

satisfied by functions invariant under X_f and X_g. It is well known [21] that system (8) is compatible, provided we have the relation

$$(f, g)\big|_{E_{f,g}} = 0 \tag{9}$$

where (f, g) is the Lagrangian bracket of the functions f and g defined as a characteristic function of the contact vector field $[X_f, X_g]$, i.e. $[X_f, X_g] = X_{(f,g)}$. If relation (9) is satisfied, then it follows from $X_f(f) = f_u \cdot f$, $X_f(g) = (f, g) + f_u \cdot g$ that the vector field X_f as well as X_g is tangent to the submanifold $E_{f,g}$.

Suppose h is a smooth function defined on $J^1(R^3)$, which annihilates on $E_{f,g}$, then h can be presented in the form $h = af + bg$ with a, b being some functions on $J^1(R^3)$. Therefore, the vector field $X_h = aX_f + bX_g + fX_a + gX_b - h\partial u$ restricted to $E_{f,g}$ is truncated to the modular combination $X_h = aX_f + bX_g$ of the vector fields X_f and X_g. This implies that X_h is tangent to $E_{f,g}$. These considerations yield the following assertion.

Theorem 2. Let X_f and X_g be vector fields whose characteristic functions are functionally independent and satisfy relation (9); then the restrictions of the vector fields $X_f^{(k)}$ and $X_g^{(k)}$ to $E_{f,g}^{(k)}$ generate an involutive module $\mathfrak{g}$ of vector fields on $E_{f,g}^{(k)}$. If the relation

$$\operatorname{rank}\begin{Vmatrix} \partial f/\partial p_t & \partial f/\partial p_x \\ \partial g/\partial p_t & \partial g/\partial p_x \end{Vmatrix} = 2$$

is satisfied, then the functions f and g may be taken in the form $f = -p_t + a(t, x, u)$, $g = -p_x + b(t, x, u)$.

Let us say that the vector fields X_f and X_g, satisfying the conditions of theorem 2, generate the two-dimensional module of partial symmetries of equation (1) if $X_f^{(k)}$ and $X_g^{(k)}$ are tangent to the intersection $E_{f,g}^{(k)} \cap E_\Delta$.

3. Weak symmetries

In [19, 20] the concept of non-classical weak symmetry of differential equations was introduced. We propose to treat infinitesimal weak symmetries corresponding to one-parameter groups as follows. Consider the contact vector field X_f, the function $\Gamma(t, x, u, p) = X_f^{(k)}(\Delta)(t, x, u, p)$, and the system W of differential equations

$$\Delta = 0 \qquad \Gamma = 0 \qquad f = 0. \tag{10}$$

Definition. The vector field X_f is an infinitesimal weak symmetry of equation (1) if

(i) X_f is a classical infinitesimal symmetry of system (10),

(ii) system (10) is compatible.

Property (i) can be reformulated by saying that X_f is a partial infinitesimal symmetry of the system $\Delta = 0$, $\Gamma = 0$. Property (ii) means that system (10) implies no extra compatibility conditions that may arise by cross differentiation of the equations of system (10) and their differential consequences. Since $\Gamma = X_f^{(k)}(\Delta)$ and $X_f(f) = f_u \cdot f$, the criterion that X_f is tangent to W takes the form

$$X(\Gamma)\big|_W = 0. \tag{11}$$

An example of infinitesimal weak symmetry of the Boussinesq equation given in [19] admits a natural intepretation in the framework of the above definition.

Example. Consider the Boussinesq equation

$$\Delta \equiv \gamma u_{xxxx} + \beta(u^2)_{xx} + u_{xx} - u_{tt} = 0 \tag{12}$$

and the contact vector field X_f with characteristic function $f = -p_t + 2ctp_x$, where c is a real parameter:

$$X_f = \partial t - 2ct\partial x + 2cp_x\partial p_t. \tag{13}$$

It is easy to see that $\Gamma \equiv X_f^{(4)}(\Delta) = -4cp_{tx}$, so X_f is not a classical symmetry of equation (12) because the relation $p_{tx} = 0$ does not follow from (12). Since equation (3) for the functions invariant under X_f is $p_t - 2ctp_x = 0$, and since its prolongation to $J^2(R^3)$ is determined by

$$p_t - 2ctp_x = 0 \qquad p_{tt} - 2ctp_{tx} = 2cp_x \qquad p_{tx} - 2cp_{xx} = 0 \tag{14}$$

the relation $p_{tx} = 0$ is not a consequence of (12) and (14). This means that X_f is not a partial symmetry of the Boussinesq equation. At the same time, $X_f^{(2)}(\Gamma) = -8c^2p_{xx}$. Therefore, the latter function vanishes on the submanifold determined by (12), (14) and $\Gamma = 0$. So the vector field considered satisfies the property (i) of our definition of the infinitesimal weak symmetry for (12).

Analysing the compatibility of system (10) in the case of the Bousssinesq equation, we immediately get from equations (12), from $\Gamma \equiv -4cp_{tx} = 0$, and from (14) that solutions of (12) invariant under the vector field (13) satisfy the following compatible system of equations:

$$u_x = \frac{c}{\beta} \qquad u_t = \frac{2c^2t}{\beta}. \tag{15}$$

Each of these equations can be treated as a reduced equation. Indeed, from the first equation of (15) one can find that $u = (c/\beta)x + g(t)$, and after substituting this expression into the second equation of system (15) one obtains the reduced equation for the function $g(t)$:

$$\dot{g}(t) = \frac{2c^2 t}{\beta}.$$

Thus we obtain the solution $u = (c/\beta)x + (c^2t^2)/\beta + c_0$ of the Boussinesq equation.

Theorem 3. Suppose that the vector field X_f is an infinitesimal weak symmetry of equation (1) and is neither a classical nor a partial symmetry of that equation, and suppose that its characteristic function satisfies the relation: rank $\|f_{p_t}, f_{p_x}\| = 1$. Suppose also that X_f-invariant solutions generate at least a one-parameter family of solutions. Then there exists a two-dimensional module $\mathfrak{g}$ of partial symmetries of equation (1) such that each X_f-invariant solution is $\mathfrak{g}$-invariant; besides, the relation $E_{\mathfrak{g}}^{(k)} \cap E_\Delta = E_{\mathfrak{g}}^{(k)}$ is valid, where $E_{\mathfrak{g}}$ is the submanifold of $\mathfrak{g}$-invariant solutions given by equations of the form (8).

Proof. To begin with, consider second-order equation (1). In the case of two independent variables, one can express one of the first derivatives, say p_t, through the variables t, x, u, p_x from equation (3). Quite similarly, the variables p_{tt}, p_{tx} are functions of the variables t, x, u, p_x, p_{xx}. If the equation $\Gamma = 0$ is a first-order differential equation, then all derivatives can be found in terms of t, x and u. Specifically,

$$p_t = a(t, x, u) \qquad p_x = b(t, x, u). \tag{16}$$

If the equation $\Gamma = 0$ is a second-order equation, then we come to (16) using both equations $\Delta = 0$ and $\Gamma = 0$. This means that system (10) is equivalent to a first-order system as far as the families of their solutions are concerned. Since X_f-invariant solutions form at least a one-parameter family, only two of the three equations (10) are independent. Thus, we may consider only equations (16). The compatibility condition for (16) is (9); consequently, the characteristic functions $g = -p_t + a(t, x, u)$ and $h = -p_x + b(t, x, u)$ generate a two-dimensional module of partial symmetries of equation (1), since, in the case considered, $E_{\mathfrak{g}}^{(2)} \cap E_\Delta = E_{\mathfrak{g}}^{(2)}$.

In the general case of kth-order ($k \geqslant 3$) differential equations for each $|\sigma| = 1, \ldots, k$, all partial derivatives p_σ except one, say p_{σ_0}, can be expressed through the coordinates of the space $J^{|\sigma|-1}(R^3)$ and through p_{σ_0} with the help of the equations of the submanifold $E_f^{(k)}$. Then the equations $\Delta = 0$, $\Gamma = 0$ allow the order of system (10) to be diminished progressively by at least two at each step. After that, the proof is continued as in the case $k = 2$. □

The theorem just proved means that obtaining vector fields of weak symmetries is equivalent in general to finding the two-dimensional modules of the partial symmetries. Our calculations show that the latter problem is essentially simpler than the first one.

4. Two-dimensional modules of partial symmetries for nonlinear heat equations

Consider the problem of finding the two-dimensional modules $g = L(X_f, X_g)$ of partial symmetries for the family of nonlinear heat equations

$$u_t = \big(\alpha(u)u_x\big)_x + \beta(u). \tag{17}$$

By theorem 2, the characteristic functions of the basic vector fields X_f and X_g can be taken in the form $f = -p_t + a(t,x,u)$ and $g = -p_x + b(t,x,u)$, and equations (8) can be transformed to (16). The functions $a(t,x,u)$ and $b(t,x,u)$ in (16) must satisfy the compatibility condition

$$a_x + a_u b = b_t + b_u a. \tag{18}$$

Besides, in order that (16) and (17) admit a one-parameter family of solutions, equation (17) must be a differential corollary of (16). This implies the following relation:

$$a = \alpha(b_x + b_u b) + \alpha' b^2 + \beta. \tag{19}$$

If we substitute the function $a(t,x,u)$ given by (19) into equation (18), we obtain the equation

$$b_t = \alpha(b_{xx} + 2bb_{ux} + b^2 b_{uu}) + \alpha'(3bb_x + 2b^2 b_u) + \alpha'' b^3 + b\beta' - \beta b_u \tag{20}$$

for the function $b(t,x,u)$.

Consider the following problem: for what functions $\alpha(u)$ and $\beta(u)$ does equation (20) admit solutions of the form $b(t,x,u) = \theta(t)H(u)$? To answer this question, substitute the function $b(t,u)$ given by the latter relation into (20). We get

$$\dot\theta(t) = \theta(t)^3 H(u)\big(\alpha(u)H(u)\big)'' + \theta(t)H(u)\big(\beta(u)/H(u)\big)'$$

which yields the relations

$$H(u)\big(\alpha(u)H(u)\big)'' = \lambda \qquad H(u)\big(\beta(u)/H(u)\big)' = \mu \tag{21}$$

with λ and μ constants. Given the function $H(u)$, equations (21) can be solved for $\alpha(u)$ and $\beta(u)$. The function $b(t,u)$ taken as the product $\theta(t)H(u)$ implies that the invariant solutions $u(t,x)$ of equation (17) can be written in the form

$$u(t,x) = F\big(\theta(t)x + \phi(t)\big). \tag{22}$$

in view of the second equation (16). In (22) the function $\theta(t)$ is a solution of the ordinary differential equation $\dot\theta = \lambda\theta^3 + \mu\theta$, which can be integrated explicitly, and the function $\phi(t)$ is a solution of the equation obtained after substituting (22) into either (17) or the first equation (16) with $a(t,u)$ given by (19). For example, for $H(u) = u^{-1}$ and $\alpha(u) = u^2 + u$, $\beta(u) = u/2$, we get the one-parameter family of invariant solutions:

$$u(t,x) = \sqrt{2(e^t x + e^{2t} + ce^t)}\,.$$

5. Weak symmetries of the nonlinear heat equation

Returning to the general discussion of weak symmetries, it is interesting to consider an example of the infinitesimal weak symmetry that admits a unique invariant solution and thereby does not fall under theorem 3. In what follows, the exact solutions of nonlinear heat equation (23) first obtained in [22] are interpreted as invariant under weak symmetries. Consider the equation

$$u_t = u_{xx} + u_x^2 + u^2 \tag{23}$$

and its infinitesimal weak symmetry with a characteristic function $f = -p_x + b(t, x)$, where the function $b(t, x)$ needs to be defined. Since the equations of the intersection $E_\Delta \cap E_f^{(2)}$ are equivalent to the equations

$$\widetilde{\Delta} \equiv -p_t + b_x + b^2 + u^2 = 0 \qquad p_x = b \qquad p_{tx} = b_t \qquad p_{xx} = b_x$$

the following formula is valid: $\Gamma \equiv X_f(\widetilde{\Delta}) = b_t + b_{xx} + 2bb_x + 2ub$. Therefore, if the function $b(t, x)$ is fixed, a unique invariant solution $u(t, x)$ is obtained from the equation $\Gamma = 0$:

$$u(t, x) = \frac{b_t - b_{xx} - 2bb_x}{2b}. \tag{24}$$

From the above calculations we can conclude that system (10) in the case considered takes the form:

$$p_t = b_x + b^2 + u^2 \qquad p_x = b(t, x) \qquad u = \frac{b_t - b_{xx} - 2bb_x}{2b}. \tag{25}$$

The compatibility conditions for system (25) are evident:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{b_t - b_{xx} - 2bb_x}{2b}\right) = b \\ &\frac{\partial}{\partial t}\left(\frac{b_t - b_{xx} - 2bb_x}{2b}\right) = b_x + b^2 + \left(\frac{b_t - b_{xx} - 2bb_x}{2b}\right)^2. \end{aligned} \tag{26}$$

Equations (26) admit separation of variables. Precisely, the first equation (26) is satisfied if $b(t, x) = \phi(t)\sin x$ with $\phi(t)$ arbitrary. Hence the second equation (25) implies the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot\phi + \phi}{2\phi}\right) = \left(\frac{\dot\phi + \phi}{2\phi}\right)^2 + \phi^2 \tag{27}$$

for the function $\phi(t)$.

After the function $\phi(t)$ is found from (27), we get the infinitesimal weak symmetry X_f with $f(t, x, u, p_t, p_x) = -p_x + \phi(t)\sin x$ and the solution of equation (23) invariant under X_f is given by

$$u(t, x) = \frac{\dot\phi + \phi}{2\phi} - \phi\sin x. \tag{28}$$

Galaktionov obtained this solution by directly applying the method of generalized separation of variables in the form $u(t, x) = \theta(t) - \phi(t)\sin x$ to equation (23). Note that this example is analysed in [23] from the point of view of differential constraints.

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